Elliptic Curve Cryptography

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Outline

1. Cryptography
2. Elliptic curves
3. Elliptic curves in practice
4. Elliptic curves in theory
5. Elliptic curves at QUT

Cryptography

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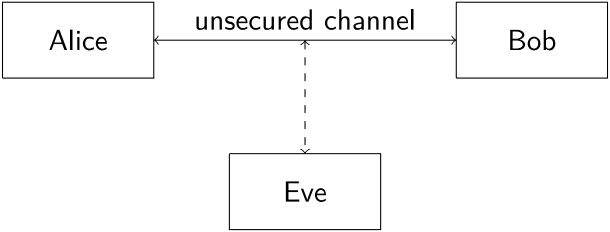
Cryptography aims to provide **confidentiality** and **integrity** of communications.



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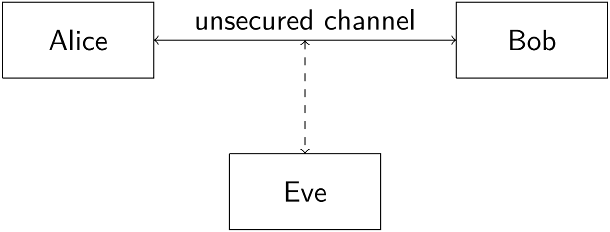

- **Symmetric key cryptography:** Alice and Bob share a secret key k that Eve does not know. (Fast!)
- **Public key cryptography:** Alice and Bob have each other's public keys pk_A and pk_B but no shared secrets. (Slow!)

Public key cryptography

Alice generates a pair of related keys:

- pk_A: her **public key**, which she gives to anyone who wants to communicate with her
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- **key agreement**: Alice and Bob compute a shared key k that they can use with symmetric encryption

Cryptography on the web

Suppose Alice wants to securely send her credit card number to `bob.com`.

1. Alice obtains a true copy of the public key pk_B for `bob.com`.
2. Alice and Bob run a key agreement protocol to get a shared secret k.
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The protocol that implements this is the **Secure Sockets Layer (SSL)** protocol, also known as the **Transport Layer Security (TLS)** protocol, which is the "s" in "`https`".

Modular arithmetic

$a \mod n$

- Let n be a positive integer and a be a non-negative integer.
- $a \mod n$ is the remainder when a is divided by n.
- Example: $12 \mod 5 = 2$

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primitive root $\bmod n$

- Let g and n be positive integers.
- g is a **primitive root** $\bmod n$ if $g^{n-1} \mod n = 1$ but $g^i \mod n \neq 1$ for any $1 \leq i < n-1$.
- Example:

g	g^2	g^3	g^4	g^5	$g^6 \mod 7$
2	4	$8 = 1$	2	4	1
3	$9 = 2$	6	$18 = 4$	$12 = 5$	$15 = 1$

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If Eve does not interfere:

- Alice computes $k = B^a = (g^b)^a = g^{ba} \mod p$
- Bob computes $k' = A^b = (g^a)^b = g^{ab} = g^{ba} \mod p$

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2. Is there any other way of computing k?
   - Not that we know of. But to prove that's the case is an open problem.

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Let p be a prime and $p-1$ be divisible by a suitably large prime. Then the **best known (classical) algorithm** for computing discrete logarithms takes

$$L_p = \exp\left(\sqrt[3]{\frac{64}{9}}(\ln p)^{1/3}(\ln\ln p)^{2/3}\right)$$

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operations per year:

$$10^6 \text{ PCs} \times 365 \text{ days} \times 24 \text{ hrs} \times 60 \text{ mins} \times 60 \text{ secs} \times 3 \times 10^9 \text{ ops} = 2^{76.3}$$

Diffie–Hellman key exchange in a group

- A **group** is a set G along with an operation $\cdot$ which is closed, associative, has an identity element, and inverses exist.
  Example: $\mathbb{Q} \setminus \{0\}$ under multiplication.
- An **abelian group** is a group where the operation is commutative.
- A group has **order** q if there exists an element $g \in G$ such that $\{g^0, g^1, \ldots, g^{q-1}\} = G$; g is called a **generator**

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$k \leftarrow B^a (= g^{ba})$		$k' \leftarrow A^b (= g^{ab})$

Elliptic curves

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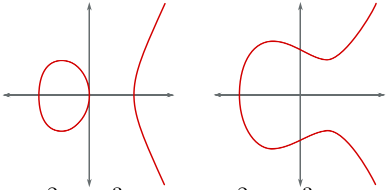
An **elliptic curve over $\mathbb{R}$** is the set of real points satisfying an equation of the form

$$y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{R}$ and $4a^3 + 27b^2 \neq 0$.



$y^2 = x^3 - x$ $\quad$ $y^2 = x^3 - x + 1$

Elliptic curve points as a group

We will construct a group consisting of the points of an elliptic curve under the operation of **point addition**.

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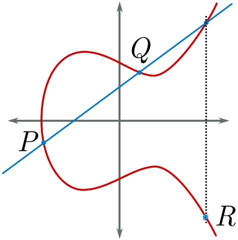
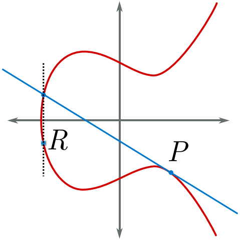


$P + Q = R$



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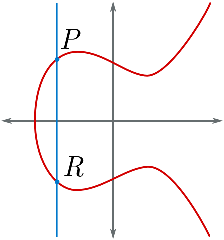

$P + R = O$

From the geometric intuition, we can easily compute algebraic formulas for point addition, point doubling, and point negation.

Elliptic curve scalar–point multiplication

Having defined point addition and point doubling, we can define **scalar–point multiplication**:

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Input: $k = (k_{\ell-1}, \ldots, k_1, k_0)_2,\ P$

1. $Q \leftarrow O$
2. for i from $\ell - 1$ to 0 do:
   - 2.1 $Q \leftarrow 2Q$
   - 2.2 if $k_i = 1$ then $Q \leftarrow Q + P$

Output: $Q = kP$

Elliptic curves over prime fields

Let p be a prime. An **elliptic curve over $\mathbb{Z}_p$** is the set of integer points $\mod p$ satisfying an equation of the form

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ECDH can achieve the **same level of security** with much **smaller values**.
Smaller values $\implies$ faster computation.

Elliptic curves in practice

ECC on the Internet

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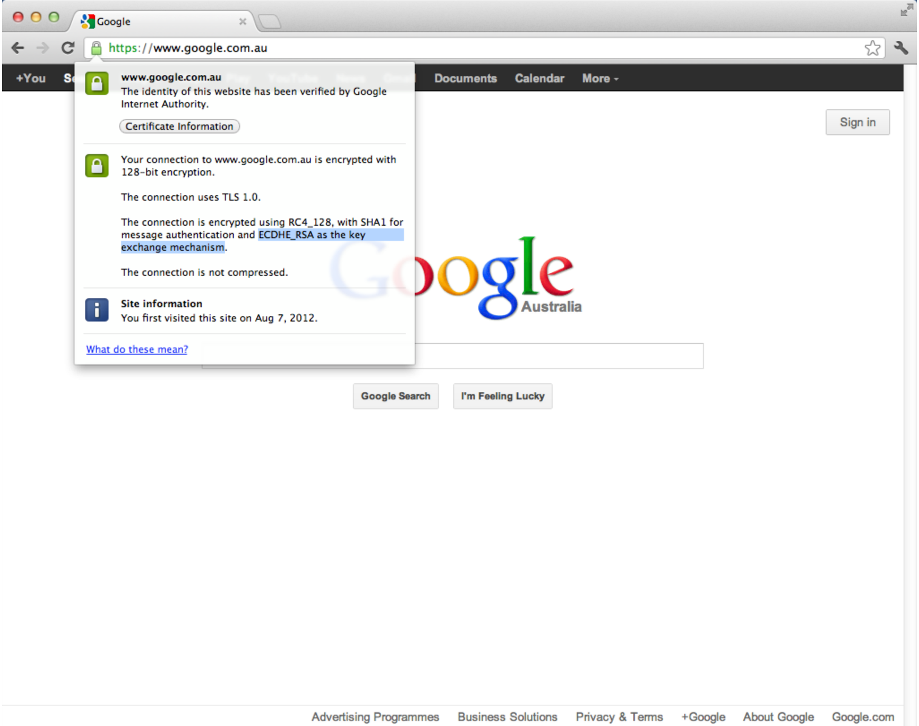
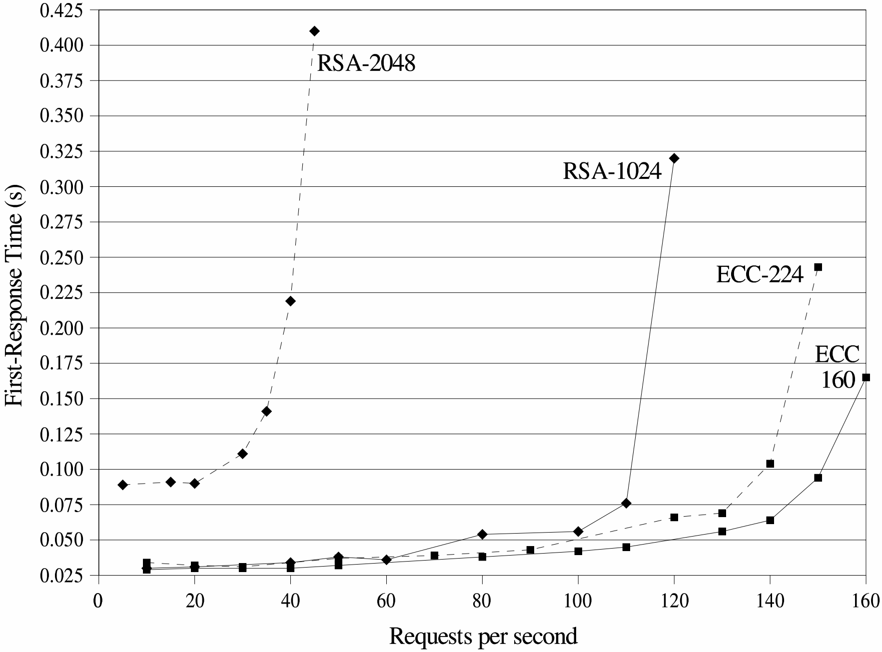
- Faster computation.
- Better security compared to existing RSA ciphersuites.
- **Forward security**: If Google's long term public key gets compromised later, your current encryptions can't be broken.


Google
https://www.google.com.au
+You
Documents
Calendar
More
www.google.com.au
The identity of this website has been verified by Google Internet Authority.
Certificate Information
Your connection to www.google.com.au is encrypted with 128-bit encryption.
The connection uses TLS 1.0.
The connection is encrypted using RC4_128, with SHA1 for message authentication and ECDHE_RSA as the key exchange mechanism.
The connection is not compressed.
Site information
You first visited this site on Aug 7, 2012.
What do these mean?
Sign in
Google
Australia
Google Search
I'm Feeling Lucky
Advertising Programmes
Business Solutions
Privacy & Terms
+Google
About Google
Google.com



First-Response Time (s)
0.425
0.400
0.375
0.350
0.325
0.300
0.275
0.250
0.225
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0
20
40
60
80
100
120
140
160
Requests per second
RSA-2048
RSA-1024
ECC-224
ECC
160

Side-channel attacks on point multiplication

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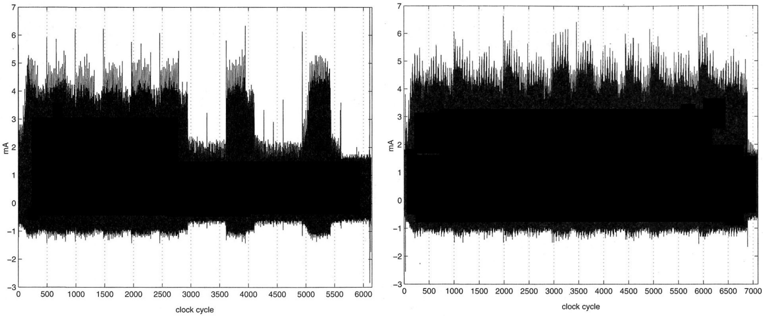

Figure : Point doubling and point addition

Side-channel attacks on point multiplication

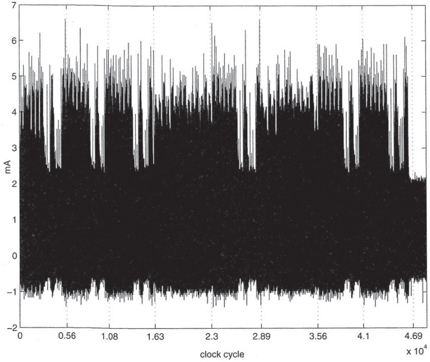

Figure : Point multiplication

Side-channel attacks on point multiplication

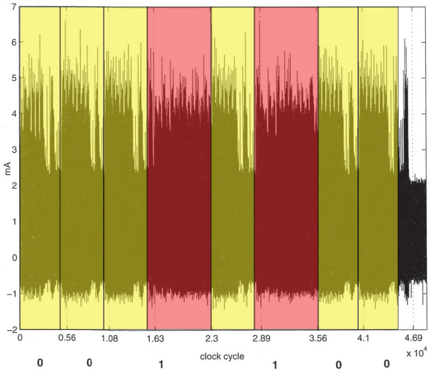

Figure : Point multiplication with additions and doublings identified

Elliptic curves in theory

Elliptic curve pairings

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Pairings can be used to construct many cryptographic protocols:

- **3-party Diffie–Hellman key exchange**:
  Alice $A = g^a$, Bob $B = g^b$, Charlie $C = g^c$
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- **identity-based encryption**:
  Instead of having to get Bob's public key, Alice can encrypt a message based on Bob's identity, such as `bob@gmail.com`.

Fermat's Last Theorem

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  would have unusual properties (violate the modularity theorem).
- **Wiles (1995).** Proof of modularity theorem and Fermat's Last Theorem. 100+ pages.

Elliptic curve cryptography at QUT

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Research:

- early implementations of ECC
- fast algorithms for ECC and pairings
- side-channel-resistant algorithms for ECC
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Teaching:

- **MAB461 Discrete Mathematics**:
  modular arithmetic, number theory, RSA public key cryptography
- **MAN778 Applications of Discrete Mathematics**:
  advanced number theory, group theory, Diffie–Hellman, introduction to elliptic curves, provable security
- **INN355 Cryptology and Protocols**:
  symmetric and public key cryptography
- **INN652 Advanced Cryptology**: elliptic curve cryptography